6 Circle

Let's Study

- Equation of a circle and its different forms
- Equation of Tangent to a circle
- Condition for tangency
- Director circle

Let's Recall

- Properties of chords and tangents of a circle.
- The angle inscribed in a semicircle is a right angle.
- Product of slopes of perpendicular lines is -1.
- Slopes of parallel lines are equal.

A circle is a set of all points in a plane which are equidistant from a fixed point in the plane.

The fixed point is called the centre of the circle and the distance from the centre to any point on the circle is called the radius of the circle.

6.1 Different forms of equation of a circle

(1) Standard form : In Fig. 6.1, the origin, O is the centre of the circle. P(x, y) is any point on the circle. The radius of circle is r.

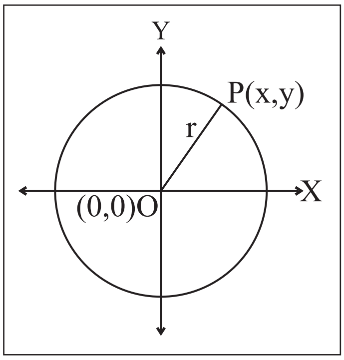


Fig. 6.1

$\therefore$ OP = r.

By distance formula

$OP^2 = (x-0)^2 + (y-0)^2$

$\therefore$ we get $r^2 = x^2 + y^2$

$\therefore x^2 + y^2 = r^2$.

This is the standard equation of a circle.

(2) Centre-radius form :

In Fig. 6.2, C(h, k) is the centre and r is the radius of the circle. P(x, y) is any point on the circle.

$\therefore$ CP = r

Also,

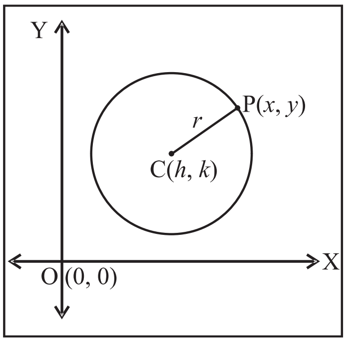


Fig. 6.2

$CP = \sqrt{(x-h)^2 + (y-k)^2}$ $\therefore$ $r^2 = (x - h)^2 + (y - k)^2$

$(x - h)^2 + (y - k)^2 = r^2$ is the centre-radius form of equation of a circle.

(3) Diameter Form : In the Fig. 6.3, C is the centre of the circle. $A(x_1, y_1)$, $B(x_2, y_2)$ are the end points of a diameter of the circle. P(x, y) is any point on the circle. Angle inscribed in a semi circle is a right angle; hence, m∡ APB = 90°; that is AP ⊥ BP.

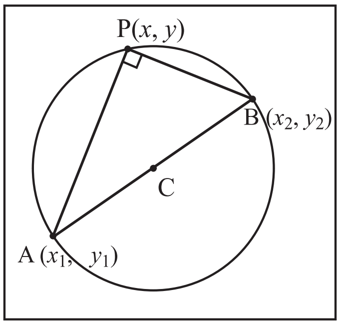


Fig. 6.3

Slope of AP = $\frac{y-y_1}{x-x_1}$ and slope of BP = $\frac{y-y_2}{x-x_2}$,

As AP ⊥ BP, product of their slopes is –1.

$\therefore \frac{y-y_1}{x-x_1} \times \frac{y-y_2}{x-x_2} = -1$

$(y - y_1)(y - y_2) = -(x - x_1)(x - x_2)$

$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$

That is, $(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$

This is called the diameter form of the equation of circle, where (x_1, y_1) and (x_2, y_2) are endpoints of diameter of the circle.

SOLVED EXAMPLES

Ex.1 Find the equation of a circle with centre at origin and radius 3.

Solution : Standard equation of a circle is

$x^2 + y^2 = r^2$ here $r = 3$

$\therefore$ $x^2 + y^2 = 3^2$

$x^2 + y^2 = 9$ is the equation of circle

Ex.2 Find the equation of a circle whose centre is (–3, 1) and which pass through the point (5, 2).

Solution : Centre C = (–3, 1),

Circle passes through the point P(5, 2).

By distance formula,

$r^2 = CP^2 = (5 + 3)^2 + (2 - 1)^2$

$= 8^2 + 1^2 = 64 + 1 = 65$

$\therefore$ the equation of the circle is

$(x + 3)^2 + (y - 1)^2 = 65$ (centre-radius form)

$x^2 + 6x + 9 + y^2 - 2y + 1 = 65$

$x^2 + y^2 + 6x - 2y + 10 - 65 = 0$

$x^2 + y^2 + 6x - 2y - 55 = 0$

Ex.3 Find the equation of the circle with A(2, –3) and B(–3, 5) as end points of its diameter.

Solution : By using the diameter form;

A (2, –3) $\equiv (x_1, y_1)$ and B(–3, 5) $\equiv (x_2, y_2)$ are the co-ordinates of the end points of a diameter of the circle.

$\therefore$ by the diameter form, equation of the circle is

$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$

$\therefore (x - 2)(x + 3) + (y + 3)(y - 5) = 0$

$\therefore x^2 + x - 6 + y^2 - 2y - 15 = 0$

$\therefore x^2 + y^2 + x - 2y - 21 = 0$

Ex.4 Find the equation of circle touching the Y-axis at point (0, 3) and whose Centre is at (–3, 3).

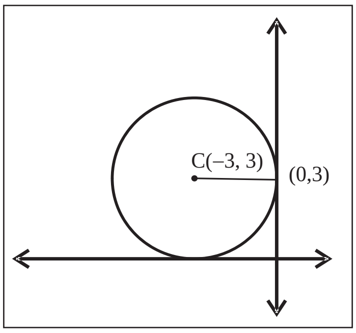


Fig. 6.4

Solution : The circle touches the Y-axis at point (0,3), and the centre is (–3,3) we get radius $r = 3$

By using centre radius form;

$(x - h)^2 + (y - k)^2 = r^2$

$(x + 3)^2 + (y - 3)^2 = 9$

$x^2 + 6x + 9 + y^2 - 6y + 9 = 9$

$\therefore$ $x^2 + y^2 + 6x - 6y + 9 = 0$ is the equation of the circle.

Ex.5 Find the equation of the circle whose centre is at (3,–4) and the line $3x–4y–5 = 0$ cuts the circle at A and B; where $l(AB) = 6$.

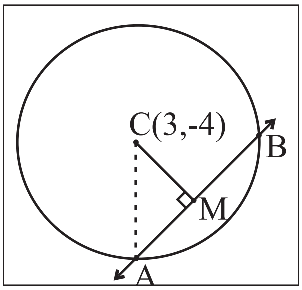


Fig. 6.5

Solution : centre of the circle $C(h,k) = C(3,-4)$

$3x - 4y - 5 = 0$ cuts the circle at A and B.

$l(AB) = 6$

$CM \perp AB$

$\therefore AM = BM = 3$

CM = Length of perpendicular from centre on the line

$$= \left|\frac{3(3) - 4(-4) - 5}{\sqrt{(3)^2 + (-4)^2}}\right|$$

$$= \left|\frac{9 + 16 - 5}{\sqrt{9 + 16}}\right|$$

$$= \left|\frac{20}{5}\right| = 4$$

From right angled triangle AMC

$CA^2 = CM^2 + AM^2$

$= (4)^2 + (3)^2$

$= 16 + 9 = 25$

CA = radius of the circle = 5

By centre radius form equation of the circle.

$(x-3)^2 + (y+4)^2 = 5^2$

$x^2-6x+9+y^2+8y+16 = 25$

$x^2+y^2-6x+8y = 0$

EXERCISE 6.1

1. Find the equation of the circle with
   (i) Centre at origin and radius 4.
   (ii) Centre at (–3, –2) and radius 6.
   (iii) Centre at (2, –3) and radius 5.
   (iv) Centre at (–3, –3) passing through point (–3, –6)

2. Find the centre and radius of the circle.
   (i) $x^2 + y^2 = 25$ (ii) $(x-5)^2 + (y-3)^2 = 20$
   (iii) $\left(x-\frac{1}{2}\right)^2 + \left(y+\frac{1}{3}\right)^2 = \frac{1}{36}$

3. Find the equation of the circle with centre
   (i) At (a, b) touching the Y-axis
   (ii) At (-2, 3) touching the X- axis
   (iii) on the X-axis and passing through the origin having radius 4.
   (iv) at (3,1) and touching the line $8x - 15y + 25 = 0$

4. Find the equation circle if the equations of two diameters are $2x + y = 6$ and $3x + 2y = 4$. When radius of circle is 9.

5. If $y = 2x$ is a chord of circle $x^2 + y^2 - 10x = 0$, find the equation of circle with this chord as diametre.

6. Find the equation of a circle with radius 4 units and touching both the co-ordinate axes having centre in third quadrant.

7) Find the equation of circle (a) passing through the origin and having intercepts 4 and –5 on the co-ordinate axes.

8) Find the equation of a circle passing through the points (1,–4), (5,2) and having its centre on the line $x-2y+9 = 0$.

Activity :

(1) Construct a circle in fourth quadrant having radius 3 and touching Y-axis. How many such circles can be drawn?

(2) Construct a circle whose equation is $x^2 + y^2 - 4x + 6y - 12 = 0$. Find the area of the circle.

6.2 General equation of a circle :

The general equation of a circle is of the form $x^2 + y^2 + 2gx + 2fy + c = 0$, if $g^2 + f^2 - c > 0$

The centre-radius form of equation of a circle is

$(x-h)^2 + (y-k)^2 = r^2$

i.e $x^2 - 2hx + h^2 + y^2 - 2ky + k^2 = r^2$

i.e $x^2 + y^2 - 2hx - 2ky + (h^2 + k^2 - r^2) = 0$

If this is the same as equation $x^2 + y^2 + 2gx + 2fy + c = 0$, then comparing the coefficients

$2g = -2h$, $2f = -2k$ and $c = h^2 + k^2 - r^2$.

$\therefore$ $(h, k) \equiv (-g, -f)$ is the center and

$r^2 = h^2 + k^2 - c$ i.e. $r = \sqrt{g^2+f^2-c}$ is the radius.

Thus,

> the general equation of a circle is $x^2 + y^2 + 2g\,x + 2\,fy + c = 0$ whose centre is $(-g, -f)$ and radius is $\sqrt{g^2 + f^2 - c}$.

Activity :

Consider the general equation of a circle

$x^2 + y^2 + 2gx + 2\,fy + c = 0$

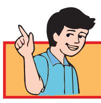
$x^2 + 2gx + \square + y^2 + 2\,fy + \square = \square + \square - c$

$\therefore\ (x + \square)^2 + (y + \square)^2 = g^2 + f^2 - c.$

$\therefore\ [x - (\quad)]^2 + [y - (\quad)]^2 = (\qquad)^2$

(use centre radius form of equation of the circle).

Therefore centre of the circle is $(-g, -f)$ and radius is $\sqrt{g^2 + f^2 - c}$.



Let's Remember

(1) If $g^2 + f^2 - c > 0$, the equation $x^2 + y^2 + 2gx + 2fy + c = 0$ represents a circle in the xy plane.

(2) If $g^2 + f^2 - c = 0$, then the equation $x^2 + y^2 + 2gx + 2fy + c = 0$ represents a point which is true degenerate conic and is the limiting position (radius is 0).

(3) If $g^2 + f^2 - c < 0$, then the equation $x^2 + y^2 + 2gx + 2fy + c = 0$ does not represent any point in the xy plane.

Activity :

Check whether the following equations represent a circle. If, so then find its centre and radius.

a) $x^2 + y^2 - 6x - 4y + 9 = 0$

b) $x^2 + y^2 - 8x + 6y + 29 = 0$

c) $x^2 + y^2 + 7x - 5y + 15 = 0$

Lets Note :

The general equation of a circle is a second degree equation in x and y, coefficient of xy is zero, coefficient of x^2 = coefficient of y^2

SOLVED EXAMPLES

Ex. 1) Prove that $3x^2 + 3y^2 - 6x + 4y - 1 = 0$, represents a circle. Find its centre and radius.

Solution : Given equation is

$3x^2 + 3y^2 - 6x + 4y - 1 = 0$

dividing by 3, we get

$x^2 + y^2 - 2x + \frac{4y}{3} - \frac{1}{3} = 0$ comparing this with

$x^2 + y^2 + 2gx + 2fy + c = 0$

we get $2g = -2$ $\quad\quad \therefore\ g = -1$

$2f = \frac{4}{3}$ $\quad \therefore\ f = \frac{2}{3}$ and $c = \frac{-1}{3}$

$g^2 + f^2 - c = (-1)^2 + \left(\frac{2}{3}\right)^2 - \frac{-1}{3}$

$= 1 + \frac{4}{9} + \frac{1}{3} = \frac{16}{9}$

As $\frac{16}{9} > 0$ $\quad\quad g^2 + f^2 - c > 0$

$\therefore\ 3x^2 + 3y^2 - 6x + 4y - 1 = 0$ represents a circle.

$\therefore$ Centre of the circle $= (-g, -f) = (1, \frac{-2}{3})$

Radius of circle r $= \sqrt{(-g)^2 + (-f)^2 - c}$

$= \sqrt{\frac{16}{9}} = \frac{4}{3}$

Ex. 2) Find the equation of the circle passing through the points (5,–6), (1,2) and (3, –4).

Solution : Conside P = (5, –6), Q = (1, 2),

$R = (3, -4)$

Let the centre of the circle be at C(h, k)

$\therefore r = CP = CQ = CR$

= radii of the same circle

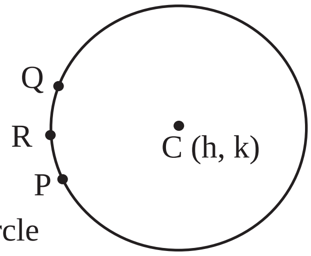


Fig. 6.6

Consider CP = CQ

$\therefore CP^2 = CQ^2$

By using distance formula,

$(h-5)^2 + (k+6)^2 = (h-1)^2 + (k-2)^2$

$\therefore h^2 - 10h + 25 + k^2 + 12k + 36$

$= h^2 - 2h + 1 + k^2 - 4k + 4$

i.e $-8h + 16k = -56$

$h - 2k = 7$ .... (I)

Now consider, CQ = CR $\quad \therefore CQ^2 = CR^2$

$(h-1)^2 + (k-2)^2 = (h-3)^2 + (k+4)^2$

$\therefore h^2 - 2h + 1 + k^2 - 4k + 4$

$= h^2 - 6h + 9 + k^2 + 8k + 16$

i.e. $4h - 12k = 20 \quad \therefore h - 3k = 5$ .... (II)

Now, subtracting (II) from I, we get,

$k = 2$

Substituting $k = 2$ in (II) we get

$h = 11 \quad \therefore C = (11, 2)$

Radius of the circle is

$r = |CP| = \sqrt{(11-5)^2 + (2+6)^2} = \sqrt{100} = 10;$

By using centre-radius form, equation of the circle is $(x-11)^2 + (y-2)^2 = 100$

$\therefore \quad x^2 - 22x + 121 + y^2 - 4y + 4 = 100$

$\therefore \quad x^2 + y^2 - 22x - 4y + 25 = 0$

Ex. 3) Show that the points (5, 5), (6, 4), (–2, 4) and (7, 1) are on the same circle; i.e. these points are concyclic.

Solution :

Let, the equation of the circle be

$x^2 + y^2 + 2gx + 2fy + c = 0$ .... (I)

The circle passing through (5, 5), (6, 4), (–2, 4)

$\therefore \quad 50 + 10g + 10f + c = 0$ .... (II)

$\therefore \quad 52 + 12g + 8f + c = 0$ .... (III)

$\therefore \quad 20 - 4g + 8f + c = 0$ .... (IV)

Now (III) – (IV) gives $\quad 32 + 16g = 0$

$\therefore 16g = -32 \quad \therefore g = -2$

Subtracting (II) from (III)

we get $2 + 2g - 2f = 0$ ..... (V)

substitute $g = -2$ in equation (V)

$2 + (-4) - 2f = 0 \quad 2 - 4 = 2f \quad \therefore f = -1$

Now substitute $g = -2$ and $f = -1$ in eqn. (IV)

$20 + c = 4(-2) - 8(1) = -8 + 8$

$20 + c = 0 \quad \therefore c = -20$

Now substituting $g = -2$, $f = -1$ and $c = -20$ in equation (I) we get,

$x^2 + y^2 - 4x - 2y - 20 = 0$ .... (VI) is the equation of the circle passing through the points (5,5), (6,4) and (–2,4).

If (7,1) satisfies equation (VI), the four points are concyclic.

L.H.S. $= (7)^2 + (1)^2 - 28 - 2 - 20$

$= 49 + 1 - 50 = 0 =$ R.H.S.

$\therefore$ Thus, the point (7, 1) satisfies the equation of circle.

$\therefore$ The given points are concyclic .

EXERCISE 6.2

(1) Find the centre and radius of each of the following.

(i) $x^2 + y^2 - 2x + 4y - 4 = 0$

(ii) $x^2 + y^2 - 6x - 8y - 24 = 0$

(iii) $4x^2 + 4y^2 - 24x - 8y - 24 = 0$

(2) Show that the equation $3x^2 + 3y^2 + 12x + 18y - 11 = 0$ represents a circle.

(3) Find the equation of the circle passing through the points (5, 7), (6, 6) and (2, –2).

(4) Show that the points (3, –2), (1, 0), (–1, –2) and (1, –4) are concyclic.

6.3 Parametric Form of a circle :

Let P(*x*, *y*) be any point on a circle with centre at O and radius r.

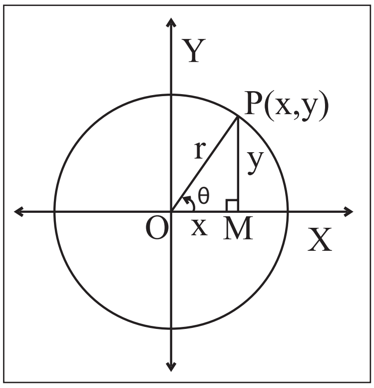


Fig. 6.7

As shown in Fig. 6.7, OP makes an angle θ with the positive direction of X-axis. Draw PM ⊥ X-axis from P.

Δ OMP is a right angled triangle,

$\therefore \cos\theta = \dfrac{OM}{OP} = \dfrac{x}{r}$; $\sin\theta = \dfrac{PM}{OP} = \dfrac{y}{r}$

$\therefore\ x = r\cos\theta$ $\therefore\ y = r\sin\theta$

> $x = r\cos\theta$ and $y = r\sin\theta$ is the parametric form of circle $x^2 + y^2 = r^2$. θ is called parameter.

Note that:

(1) The parametric form of circle $(x - h)^2 + (y - k)^2 = r^2$ is given by $x = h + r\cos\theta$ and $y = k + r\sin\theta$.

Hence co-ordinates of any point on the circle are $(h + r\cos\theta,\ k + r\sin\theta)$.

(2) Sometimes parametric form is more convenient for calculation as it contains only one variable.

6.3.1 Tangent : When a line intersects a circle in coincident points, then that line is called as a tangent of the circle and the point of intersection is called point of contact.

The equation of tangent to a standard circle $x^2 + y^2 = r^2$ at point $P(x_1, y_1)$ on it.

Given equation of a circle is $x^2 + y^2 = r^2$. The centre of the circle is at origin O(0, 0) and radius is *r*. Let P (x_1, y_1) be any point on the circle.

Slope of OP $= \dfrac{y_1 - 0}{x_1 - 0} = \dfrac{y_1}{x_1}$, if $x_1 \neq 0, y_1 \neq 0$.

A tangent is drawn to the circle at point P.

Since OP is perpendicular to tangent at point P.

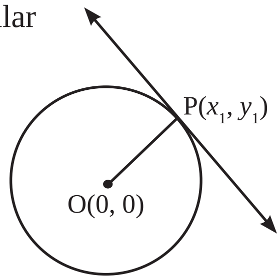


Fig. 6.8

slope of the tangent

$m = \dfrac{-x_1}{y_1}$

∴ equation of the tangent in slope point form is

$y - y_1 = m(x - x_1)$

$y - y_1 = \dfrac{-x_1}{y_1}(x - x_1)$

$yy_1 - y_1^2 = -xx_1 + x_1^2$

$xx_1 + yy_1 = x_1^2 + y_1^2$ ....(I)

As (x_1, y_1) lies on the circle, $x_1^2 + y_1^2 = r^2$, therefore equation (I) becomes $xx_1 + yy_1 = r^2$

> Thus, In general, equation of the tangent to a circle $x^2 + y^2 = r^2$ at point $P(x_1, y_1)$ is $xx_1 + yy_1 = r^2$

Follow the method given above and verify that equation of tangent to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ at (x_1, y_1) is $xx_1 + yy_1 + g(x+x_1) + f(y+y_1) + c = 0$

To find equation of tangent to the curve at (x_1, y_1) replace x^2 by $x\,x_1$, $2x$ by $(x + x_1)$, y^2 by $y\,y_1$, $2y$ by $(y + y_1)$

Equation of tangent in parametric form.

Substituting rcosθ_1 for x_1 and rsinθ_1, foy y_1, the equation of a tangent to the circle $x^2 + y^2 = r^2$ at point P $(r\cos\theta_1, r\sin\theta_1) = (x_1,y_1)$ is

$x.r\cos\theta_1 + y.r\sin\theta_1 = r^2$ i.e. $x\cos\theta_1 + y\sin\theta_1 = r$

6.3.2 Condition of tangency :

To find the condition that a line $y = mx + c$ is a tangent to the circle $x^2 + y^2 = a^2$ and also to find the point of contact,

Let the equation of the line be $y = mx + c$

$\therefore\ mx - y + c = 0$ ....(I)

Equation of a tangent to the circle $x^2 + y^2 = a^2$ at point $(x_1\ y_1)$ on it is $xx_1 + yy_1 = a^2$

i.e. $x_1\,x + y_1\,y - a^2 = 0$ ....(II)

If the line given by equation (I) is tangent to the circle then equation (I) and equation (II) represent the same (tangent) line.

Comparing the co-efficients of the like terms in these equations,

$$\frac{x_1}{m} = \frac{y_1}{-1} = \frac{-a^2}{c}$$

$$\therefore\ \frac{x_1}{m} = \frac{-a^2}{c} \text{ and } \frac{y_1}{-1} = \frac{-a^2}{c}$$

$$\therefore\ x_1 = \frac{-a^2m}{c} \text{ and } y_1 = \frac{a^2}{c}$$

But the point (x_1, y_1) lies on the circle

$$\therefore\ x_1^2 + y_1^2 = a^2$$

$$\therefore\ \left(\frac{-a^2m}{c}\right)^2 + \left(\frac{-a^2}{c}\right)^2 = a^2$$

$$\frac{a^4m^2}{c^2} + \frac{a^4}{c^2} = a^2$$

$a^2m^2 + a^2 = c^2$. i.e. $c^2 = a^2m^2 + a^2$,

which is the required condition of tangency and the point of contact $P(x_1, y_1) \equiv \left(\frac{-a^2m}{c}, \frac{a^2}{c}\right)$.

Thus a line $y = mx + c$ is a tangent to the circle $x^2 + y^2 = a^2$, if $c^2 = a^2m^2 + a^2$ i.e. $c = \pm\sqrt{a^2m^2 + a^2}$ and the point of contact is $\left(\frac{-a^2m}{c}, \frac{a^2}{c}\right)$.

Thus, there are two tangents with the same slope m, $y = mx + \sqrt{a^2(1+m^2)}$, $y = mx - \sqrt{a^2(1+m^2)}$

To check the tangency of a straight line to a circle, it is enough to show that the perpendicular from the center to the line is equal to the radius.

6.3.3 Tangents from a point to the circle

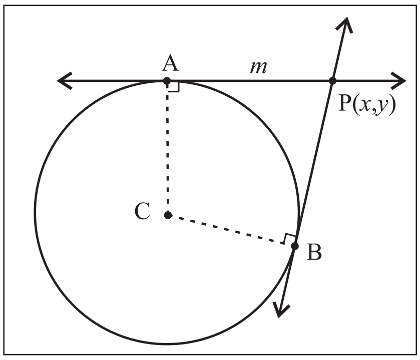


Fig. 6.9

From any point outside the circle and in the same plane, two tangents can be drawn to the circle.

Let $P(x_1,y_1)$ be a point in the plane, outside the circle.

If a tangent from P to the circle has slope m, the equation of the tangent is $y-y_1 = m(x-x_1)$ i.e. $mx-y_1-mx_1+y_1 = 0$.

The condition that this is tangent to the circle is $\left|\frac{y_1 - mx_1}{\sqrt{1+m^2}}\right|$ = a, the radius.

$\therefore (y_1 - mx_1)^2 = a^2(1+m^2)$

$\therefore x^2 - 2x_1y_1m + x_1^2m^2 = a^2 + a^2m^2$

$\therefore (x_1^2 - a^2)\ m^2 - 2x_1y_1m + (y_1^2 - a^2) = 0$ is quadratic equation in m.

It has two roots say m_1 and m_2, which are the slopes of two tangents.

Thus two tangents can be drawn to a circle from a given point in its plane.

Sum of the roots $(m_1 + m_2) = \frac{-(-2x_1\,y_1)}{(x_1^2 - a^2)}$

$= \frac{2x_1\,y_1}{x_1^2 - a^2}$

Product of the roots $(m_1\,m_2) = \frac{(y_1^2 - a^2)}{(x_1^2 - a^2)}$

6.3.4 Director Circle:

The locus of the point of intersection of perpendicular tangents to a circle

If the tangents drawn from a point P are mutually perpendicular to each other them $m_1\,m_2 = -1$ and we have,

$$\frac{y_1^2 - a^2}{x_1^2 - a^2} = -1$$

$\therefore y_1^2 - a^2 = -(x_1^2 - a^2)$

$\therefore y_1^2 - a^2 = -x_1^2 - a^2$

$\therefore x_1^2 + y_1^2 = a^2 + a^2$

$\therefore x_1^2 + y_1^2 = 2a^2.$

Which represents a circle and is called as equation of the director circle of the circle

$x^2 + y^2 = a^2.$

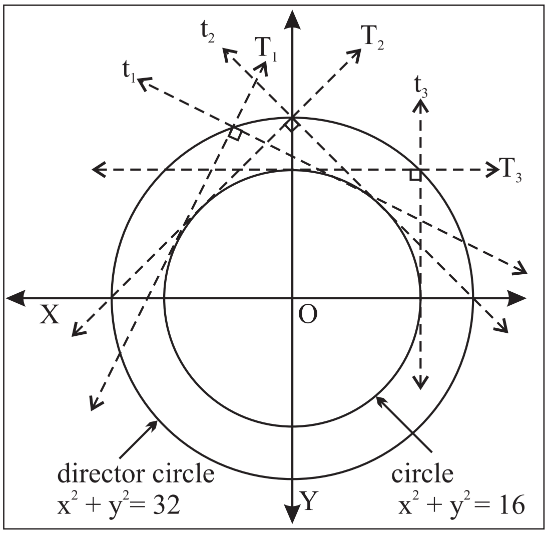


Fig. 6.10

SOLVED EXAMPLES

Ex. 1) Find the parametric equation of the circle

$x^2 + y^2 - 6x + 4y - 3 = 0$

Solution : We write the equation of the circle as

$(x^2 - 6x + 9) + (y^2 + 4y + 4) = 3 + 9 + 4$

$(x - 3)^2 + (y + 2)^2 = 16$

$(x - 3)^2 + (y + 2)^2 = (4)^2$

The parmetric equations are

$x - 3 = 4\cos\theta$ and $y + 2 = 4\sin\theta$

that is, $x = 3 + 4\cos\theta$ and $y = -2 + 4\sin\theta$

Ex. 2) Show that the line $3x - 4y + 15 = 0$ is a tangent to the circle $x^2 + y^2 = 9$. Find the point of contact.

Solution : The equation of circle is $x^2 + y^2 = 9$ ...(1)

The equation of line is $3x - 4y + 15 = 0$ ...(2)

$\therefore y = \frac{1}{4}(3x + 15)$, substitute value of y in equation (1)

$x^2 + \frac{1}{16}(3x + 15)^2 = 9$

$\therefore\ 16x^2 + 9x^2 + 90x + 225 = 144$

$\therefore\ 25x^2 + 90x + 81 = 0$

$\therefore\quad (5x + 9)^2 = 0$

$x = \frac{-9}{5}$; The roots of equation are equal.

$\therefore$ line (2) is tangent to given circle (1)

$\therefore\ y = \frac{1}{4}(3x + 15)$

$= \frac{1}{4}\left(3\left(-\frac{9}{5}\right) + 12\right)$

$= \frac{12}{5}$

$\left(-\frac{9}{3}, \frac{12}{5}\right)$ is the only point of intersection of the line and circle.

$\therefore$ The line $3x - 4y + 15 = 0$ touches the circle at $\left(-\frac{9}{3}, \frac{12}{5}\right)$

$\therefore$ point of contact = $\left(-\frac{9}{3}, \frac{12}{5}\right)$

Activity :

Equation of a circle is $x^2 + y^2 = 9$.

Its centre is at (☐,☐) and radius is ☐

Equation of a line is $3x - 4y + 15 = 0$

$\therefore\ y =$ ☐ $x +$ ☐

Comparing it with $y = mx + c$

$m =$ ☐ and $c =$ ☐

We know that, if the line $y = mx + c$ is a tangent to $x^2 + y^2 = a^2$ then $c^2 = a^2m^2 + a^2$

Hence $c^2 = (\quad)^2$

$= \frac{225}{16}$ .... (I)

Also, $c^2 = a^2m^2 + a^2$

$= 9$ ☐ $+ 9$

$= 9\left(\frac{9}{16} + 1\right)$

$= \frac{9(25)}{16}$

$= \frac{225}{16}$ ....(II)

From equations (I) and (II) we conclude that

Ex. 3) Find the equation of the tangent to the circle $x^2 + y^2 - 4x - 6y - 12 = 0$ at $(-1, -1)$

Solution : The equation of circle is

$x^2 + y^2 - 4x - 6y - 12 = 0$

It is of the type $x^2 + y^2 + 2gx + 2fy + c = 0$

$\therefore\ g = -2,\ f = -3,\ c = -12$

Let P $(-1, -1) = (x_1, y_1)$

We know that the equation of a tangent to a circle

$x^2 + y^2 + 2gx + 2fy + c = 0$ at (x_1, y_1) is

$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$

$x(-1) + y(-1) + 2(x-1) -3(y-1) -12 = 0$

$-x - y - 2x + 2 - 3y + 3 - 12 = 0$

$3x + 4y + 7 = 0$

EXERCISE 6.3

(1) Write the parametric equations of the circles

(i) $x^2 + y^2 = 9$ (ii) $x^2 + y^2 + 2x - 4y - 4 = 0$

(iii) $(x - 3)^2 + (y + 4)^2 = 25$

(2) Find the parametric representation of the circle $3x^2 + 3y^2 - 4x + 6y - 4 = 0$.

(3) Find the equation of a tangent to the circle $x^2 + y^2 - 3x + 2y = 0$ at the origin.

(4) Show that the line $7x - 3y - 1 = 0$ touches the circle $x^2 + y^2 + 5x - 7y + 4 = 0$ at point (1, 2)

(5) Find the equation of tangent to the circle $x^2 + y^2 - 4x + 3y + 2 = 0$ at the point (4, –2)

Let's Remember

1) Equation of a standard circle is $x^2 + y^2 = r^2$. Its centre is at (0, 0) and radius is r.

2) Equation of a circle in (centre-radius) form is
$(x - h)^2 + (y - k)^2 = r^2$
Its centre is at (h, k) and radius is r.

3) Equation of a circle in general form is
$x^2 + y^2 + 2gx + 2fy + c = 0$.
Its centre is at $(-g, -f)$ and radius is $\sqrt{g^2 + f^2 - c}$
If $g^2 + f^2 > c$ then the circle is real, it can be drawn in the xy plane.
If $g^2 + f^2 = c$ then the circle reduces to a point.
If $g^2 + f^2 < c$ then the circle is not real and it cannot be drawn in xy plane.

4) Equation of a standard circle $x^2 + y^2 = r^2$ in parametric form is $x = r\cos\theta$ $y = r\sin\theta$.

5) Equation of a tangent to the circle $x^2 + y^2 = r^2$ at point (x_1, y_1) on it is $xx_1 + yy_1 = r^2$ in the Cartesian form and point of contact is $\left(\frac{-r^2m}{c}, \frac{r^2}{c}\right)$. Equation of a tangent in parametric form to the circle $x^2 + y^2 = r^2$ at point $P(x_1, y_1) \equiv P(\theta_1)$, where θ_1 is the parameter, is $\cos\theta_1 x + \sin\theta_1 y = r$ is.

6) A line $y = mx + c$ is a tangent to the circle $x^2 + y^2 = r^2$ if and only if $c^2 = r^2m^2 + r^2$.

7) Equation of a tangent line in terms of slope is $y = mx \pm \sqrt{r^2m^2 + r^2}$.

8) Equation of director circle of circle $x^2 + y^2 = r^2$ is $x^2 + y^2 = 2r^2$.

MISCELLANEOUS EXERCISE - 6

(I) Choose the correct alternative.

(1) Equation of a circle which passes through (3, 6) and touches the axes is

(A) $x^2 + y^2 + 6x + 6y + 3 = 0$

(B) $x^2 + y^2 - 6x - 6y - 9 = 0$

(C) $x^2 + y^2 - 6x - 6y + 9 = 0$

(D) $x^2 + y^2 - 6x + 6y - 3 = 0$

(2) If the lines $2x - 3y = 5$ and $3x - 4y = 7$ are the diameters of a circle of area 154 sq. units, then find the equation of the circle.

(A) $x^2 + y^2 - 2x + 2y = 40$

(B) $x^2 + y^2 - 2x - 2y = 47$

(C) $x^2 + y^2 - 2x + 2y = 47$

(D) $x^2 + y^2 - 2x - 2y = 40$

(3) Find the equation of the circle which passes through the points (2, 3) and (4, 5) and the centre lies on the straight line $y - 4x + 3 = 0$.

(A) $x^2 + y^2 - 4x - 10y + 25 = 0$

(B) $x^2 + y^2 - 4x - 10y - 25 = 0$

(C) $x^2 + y^2 - 4x + 10y - 25 = 0$

(D) $x^2 + y^2 + 4x - 10y + 25 = 0$

(4) The equation of the tangent to the circle $x^2 + y^2 = 4$ which are parallel to $x + 2y + 3 = 0$ are

(A) $x - 2y = 2$ (B) $x + 2y = \pm 2\sqrt{3}$

(C) $x + 2y = \pm 2\sqrt{5}$ (D) $x - 2y = \pm 2\sqrt{5}$

(5) If the lines $3x - 4y + 4 = 0$ and $6x - 8y - 7 = 0$ are tangents to a circle, then find the radius of the circle

(A) $\frac{3}{4}$ (B) $\frac{4}{3}$ (C) $\frac{1}{4}$ (D) $\frac{7}{4}$

(6) Area of the circle centre at (1, 2) and passing through (4, 6) is

(A) 5π (B) 10π

C) 25π (D) 100π

(7) If a circle passes through the point (0, 0), (a, 0) and (0, b) then find the co-ordinates of its centre.

(A) $\left(\frac{-a}{2}, \frac{-b}{2}\right)$ (B) $\left(\frac{a}{2}, \frac{-b}{2}\right)$

(C) $\left(\frac{-a}{2}, \frac{b}{2}\right)$ (D) $\left(\frac{a}{2}, \frac{b}{2}\right)$

(8) The equation of a circle with origin as centre and passing through the vertices of an equilateral triangle whose median is of length 3a is

(A) $x^2 + y^2 = 9a^2$ (B) $x^2 + y^2 = 16a^2$

(C) $x^2 + y^2 = 4a^2$ (D) $x^2 + y^2 = a^2$

(9) A pair of tangents are drawn to a unit circle with cnetre at the origin and these tangents intersect at A enclosing an angle of 60 . The area enclosed by these tangetns and the area of the circle is

(A) $\frac{2}{\sqrt{3}} - \frac{\pi}{6}$ (B) $\sqrt{3} - \frac{\pi}{3}$

(C) $\frac{\pi}{3} - \frac{\sqrt{3}}{6}$ (D) $\sqrt{3}\left(1 - \frac{\pi}{6}\right)$

(10) The parametric equations of the circle $x^2 + y^2 + mx + my = 0$ are

(A) $x = \frac{-m}{2} + \frac{m}{\sqrt{2}}\cos\theta$, $y = \frac{-m}{2} + \frac{m}{\sqrt{2}}\sin\theta$

(B) $x = \frac{-m}{2} + \frac{m}{\sqrt{2}}\cos\theta$, $y = \frac{+m}{2} + \frac{m}{\sqrt{2}}\sin\theta$

(C) $x = 0$, $y = 0$

(D) $x = m\cos\theta$; $y = m\sin\theta$

(II) Answer the following :

Q. 1 Find the centre and radius of the circle $x^2 + y^2 - x + 2y - 3 = 0$

Q. 2 Find the centre and radius of the circle $x = 3 - 4\sin\theta$, $y = 2 - 4\cos\theta$

Q. 3 Find the equation of circle passing through the point of intersection of the lines $x + 3y = 0$ and $2x - 7y = 0$ whose centre is the point of intersection of lines $x + y + 1 = 0$ and $x - 2y + 4 = 0$.

Q. 4 Find the equation of circle which passes through the origin and cuts of chords of length 4 and 6 on the positive side of x - axis and y axis respectively.

Q. 5 Show that the points (9, 1), (7, 9), (–2, 12) and (6, 10) are concyclic.

Q. 6 The line $2x - y + 6 = 0$ meets the circle $x^2 + y^2 + 10x + 9 = 0$ at A and B. Find the equation of circle on AB as diameter.

Q. 7 Show that $x = -1$ is a tangent to circle $x^2 + y^2 - 2y = 0$ at (–1, 1).

Q. 8 Find the equation of tangent to the circle $x^2 + y^2 = 64$ at the point P $\left(\frac{2\pi}{3}\right)$

Q.9 Find the equation of locus of the point of intersection of perpendicular tangents drawn to the circle $x = 5\cos\theta$ and $y = 5\sin\theta$.

Q.10 Find the equation of the circle concentric with $x^2 + y^2 - 4x + 6y = 1$ and having radius 4 units.

Q.11 Find the lengths of the intercepts made on the co-ordinate axes, by the circle.

(i) $x^2 + y^2 - 8x + y - 20 = 0$

(ii) $x^2 + y^2 - 5x + 13y - 14 = 0$

Q.12 Show that the circles touch each other externally. Find their point of contact and the equation of their common tangent.

i) $x^2 + y^2 - 4x + 10y + 20 = 0$,
$x^2 + y^2 + 8x - 6y - 24 = 0$.

ii) $x^2 + y^2 - 4x - 10y + 19 = 0$,
$x^2 + y^2 + 2x + 8y - 23 = 0$.

Q.13 Show that the circles touch each other internally. Find their point of contact and the equation of their common tangent.

i) $x^2 + y^2 - 4x - 4y - 28 = 0$,
$x^2 + y^2 - 4x - 12 = 0$.

ii) $x^2 + y^2 + 4x - 12y + 4 = 0$,
$x^2 + y^2 - 2x - 4y + 4 = 0$.

Q.14 Find the length of the tangent segment drawn from the point (5, 3) to the circle $x^2 + y^2 + 10x - 6y - 17 = 0$.

Q.15 Find the value of k, if the length of the tangent segment from the point (8, –3) to the circle.
$x^2 + y^2 - 2x + ky - 23 = 0$ is $\sqrt{10}$.

Q.16 Find the equation of tangent to Circle $x^2 + y^2 - 6x - 4y = 0$, at the point (6, 4) on it.

Q.17 Find the equation of tangent to Circle $x^2 + y^2 = 5$, at the point (1, –2) on it.

Q.18 Find the equation of tangent to Circle $x = 5\cos\theta$. $y = 5\sin\theta$, at the point $\theta = \pi/3$ on it.

Q.19 Show that $2x + y + 6 = 0$ is a tangent to $x^2 + y^2 + 2x - 2y - 3 = 0$. Find its point of contact.

Q.20 If the tangent at (3, –4) to the circle $x^2 + y^2 = 25$ touches the circle $x^2 + y^2 + 8x - 4y + c = 0$, find c.

Q.21 Find the equations of the tangents to the circle $x^2 + y^2 = 16$ with slope –2.

Q.22 Find the equations of the tangents to the circle $x^2 + y^2 = 4$ which are parallel to $3x + 2y + 1 = 0$.

Q.23 Find the equations of the tangents to the circle $x^2 + y^2 = 36$ which are perpendicular to the line $5x + y = 2$.

Q.24 Find the equations of the tangents to the circle $x^2 + y^2 - 2x + 8y - 23 = 0$ having slope 3.

Q.25 Find the eq^n of the locus of a point, the tangents from which to the circle $x^2 + y^2 = 9$ are at right angles.

Q.26 Tangents to the circle $x^2 + y^2 = a^2$ with inclinations, θ_1 and θ_2 intersect in P. Find the locus of such that

i) $\tan\theta_1 + \tan\theta_2 = 0$ ii) $\cot\theta_1 + \cot\theta_2 = 5$
iii) $\cot\theta_1 . \cot\theta_2 = c$.

Extra Information :

1)

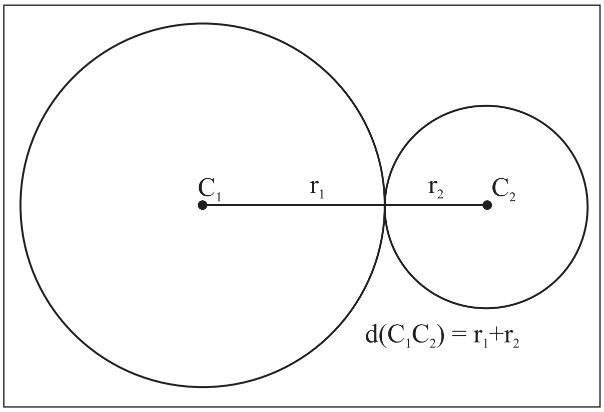


Fig. 6.10

Circles touching each other externally.
$d(c_1\ c_2) = r_1 + r_2$.
Exactly three common tangents can be drawn.

2)

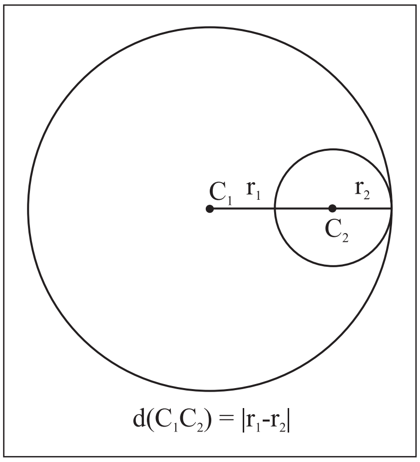


Fig. 6.11

Circles touching each other internally.

$d(c_1\ c_2) = |\ r_1 - r_2\ |$

Exactly one common tangent can be drawn.

3)

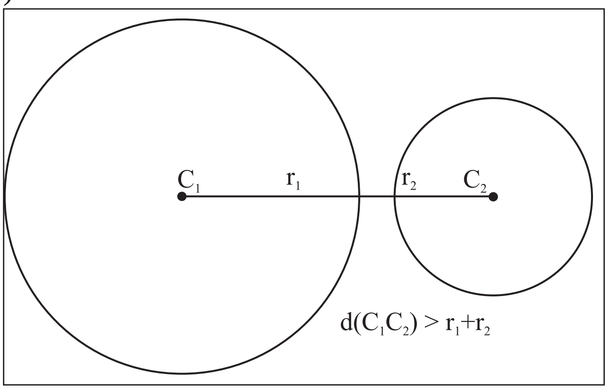


Fig. 6.12

Disjoint circles.

$|\ r_1 + r_2\ | < d(c_1\ c_2)$

Exactly four common tangents can be drawn.

4)

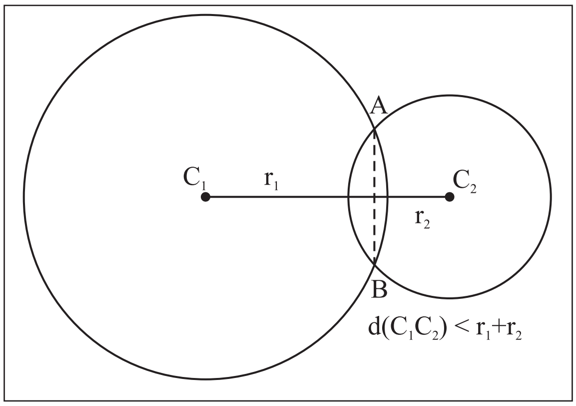


Fig. 6.13

Circles intersecting each other.

Line joining the point of intersection is the common chord also called as the radical axis.

(seg AB)

Exactly two common tangent can be drawn.

5)

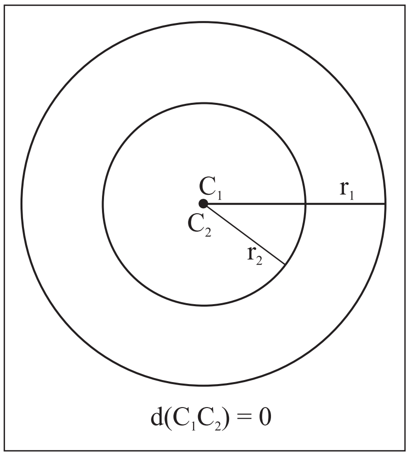


Fig. 6.14

Concentric circles

No common tangent can be drawn.